Electronic Supplement to

Study of earthquake triggering in a heterogeneous crust using a new finite element model

By Caibo Hu[1], Yijie Zhou[1], Yongen Cai[1*], Chi-Yuen Wang[2]

[1]Department of Geophysics

Peking University, Beijing, China, 100871

[2]Department of Earth and Planetary Science

University of California, Berkeley, CA94720

*: Correspondence author, Professor, E-mail: yongen@pku.edu.cn

The electronic supplement shows the principle and formulae of a new finite element model.

Suppose a region V to be studied (Figure S1) is divided into two domains, which are denoted by V_{I} outside of faults and V_{II} inside of faults, respectively. The boundary of the region consists of two portions, S_1 and S_2, with displacement $\bar{\mathbf{u}}$ prescribed on S_1 and surface traction $\mathbf{q}$ on S_2. In the standard finite element method, the formulation of the static equilibrium of the region is cast in its 'weak' form according to the principle of virtual work [Zienkiewicz and Taylor, 1986]:

$$\int_{V_{\mathrm{I}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{I}})^{\mathrm{T}} \boldsymbol{\sigma}^{\mathrm{I}} dV + \int_{V_{\mathrm{II}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{II}})^{\mathrm{T}} \boldsymbol{\sigma}^{\mathrm{II}} dV = \int_{V_{\mathrm{I}}} \tilde{\mathbf{u}}^{\mathrm{T}} \boldsymbol{\gamma}^{\mathrm{I}} dV + \int_{V_{\mathrm{II}}} \tilde{\mathbf{u}}^{\mathrm{T}} \boldsymbol{\gamma}^{\mathrm{II}} dV + \int_{S_2} \tilde{\mathbf{u}}^{\mathrm{T}} \mathbf{q} dS \qquad \text{(S1)}$$

where $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\varepsilon}}$ (expressed as vectors) denote the virtual displacement and strain, respectively, ρ is density, g is the gravitational acceleration, and $\boldsymbol{\gamma} = [0 \quad 0 \quad -\rho g]^{\mathrm{T}}$ is the body force; the superscripts I and II denote the region outside and inside the fault zone, respectively, and the superscript T denotes the matrix transpose.

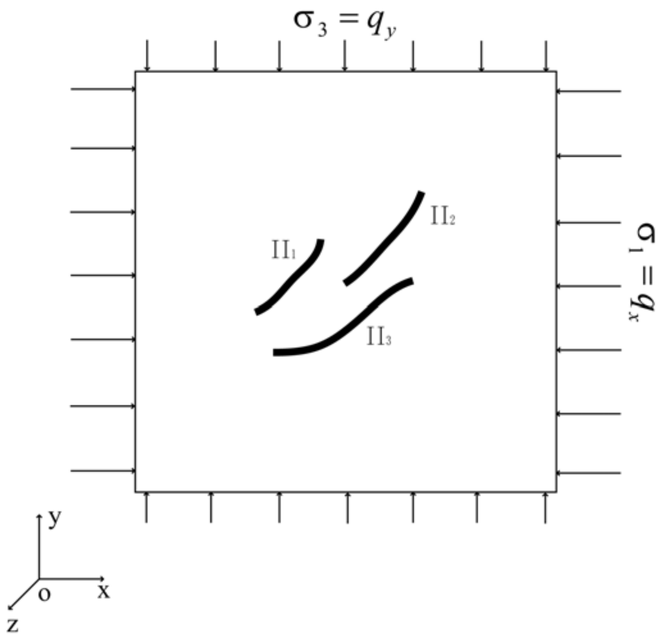

Figure S1. A sketch map to illustrate the new finite element model.

Before an earthquake we denote the material matrix and the strain and stress (expressed by vectors) by $\mathbf{D}_0^{\mathrm{I}}$, $\boldsymbol{\varepsilon}_0^{\mathrm{I}}$ and $\boldsymbol{\sigma}_0^{\mathrm{I}} = \mathbf{D}_0^{\mathrm{I}}\boldsymbol{\varepsilon}_0^{\mathrm{I}}$ outside an fault, and by $\mathbf{D}_0^{\mathrm{II}}$, $\varepsilon_0^{\mathrm{II}}$ and σ_0^{II} inside the fault. After the earthquake, the material matrix outside the faults is not changed; the strain and stress become $\boldsymbol{\varepsilon}^{\mathrm{I}} = \boldsymbol{\varepsilon}_0^{\mathrm{I}} + \Delta\boldsymbol{\varepsilon}^{\mathrm{I}}$ and $\boldsymbol{\sigma}^{\mathrm{I}} = \mathbf{D}_0^{\mathrm{I}}\boldsymbol{\varepsilon}^{\mathrm{I}}$, respectively, where $\Delta\varepsilon^{\mathrm{I}}$ is the strain increments caused by the earthquake. Inside the faults, these parameters become, respectively, $\mathbf{D}^{\mathrm{II}} = \mathbf{D}_0^{\mathrm{II}} - \Delta\mathbf{D}^{\mathrm{II}}$, $\boldsymbol{\varepsilon}^{\mathrm{II}} = \boldsymbol{\varepsilon}_0^{\mathrm{II}} + \Delta\boldsymbol{\varepsilon}^{\mathrm{II}}$ and $\boldsymbol{\sigma}^{\mathrm{II}} = \mathbf{D}^{\mathrm{II}}\boldsymbol{\varepsilon}^{\mathrm{II}}$, where $\Delta\mathbf{D}^{\mathrm{II}}$ denotes the reduction of the elastic parameters in modeling the earthquake. The inclusion of the changes in stress, strain and material property during earthquakes in the finite element modeling allows the evaluation of the evolution of stress and strain in a region during an earthquake sequence. The finite element formulation that includes these changes, as given below, is new.

Inserting $\boldsymbol{\sigma}^{\mathrm{I}} = \mathbf{D}_0^{\mathrm{I}}\boldsymbol{\varepsilon}^{\mathrm{I}} = \mathbf{D}_0^{\mathrm{I}}(\boldsymbol{\varepsilon}_0^{\mathrm{I}} + \Delta\boldsymbol{\varepsilon}^{\mathrm{I}})$ and $\boldsymbol{\sigma}^{\mathrm{II}} = \mathbf{D}^{\mathrm{II}}\boldsymbol{\varepsilon}^{\mathrm{II}} = (\mathbf{D}_0^{\mathrm{II}} - \Delta\mathbf{D}^{\mathrm{II}})(\boldsymbol{\varepsilon}_0^{\mathrm{II}} + \Delta\boldsymbol{\varepsilon}^{\mathrm{II}})$ into Equation (S1) and rearrange terms, we have

$$\begin{aligned}&\int_{V_{\mathrm{I}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{I}})^{\mathrm{T}} \mathbf{D}_0^{\mathrm{I}} \Delta\boldsymbol{\varepsilon}^{\mathrm{I}} dV + \int_{V_{\mathrm{II}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{II}})^{\mathrm{T}} \mathbf{D}^{\mathrm{II}} \Delta\boldsymbol{\varepsilon}^{\mathrm{II}} dV - \int_{V_{\mathrm{II}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{II}})^{\mathrm{T}} \Delta\mathbf{D}^{\mathrm{II}} \boldsymbol{\varepsilon}_0^{\mathrm{II}} dV \\ &= -\left(\int_{V_{\mathrm{I}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{I}})^{\mathrm{T}} \mathbf{D}_0^{\mathrm{I}} \boldsymbol{\varepsilon}_0^{\mathrm{I}} dV + \int_{V_{\mathrm{II}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{II}})^{\mathrm{T}} \mathbf{D}_0^{\mathrm{II}} \boldsymbol{\varepsilon}_0^{\mathrm{II}} dV\right) + \int_{V_{\mathrm{I}}} \tilde{\mathbf{u}}^{\mathrm{T}} \boldsymbol{\gamma}^{\mathrm{I}} dV + \int_{V_{\mathrm{II}}} \tilde{\mathbf{u}}^{\mathrm{T}} \boldsymbol{\gamma}^{\mathrm{II}} dV + \int_{S_2} \tilde{\mathbf{u}}^{\mathrm{T}} \mathbf{q} dS\end{aligned} \quad \text{(S2)}$$

Since the body force and the boundary traction may be considered constant during the earthquake sequence, the right side of equation (S2) vanishes according to (S1) and we have

$$\int_{V_{\mathrm{I}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{I}})^{\mathrm{T}} \mathbf{D}_0^{\mathrm{I}} \boldsymbol{\varepsilon}_0^{\mathrm{I}} dV + \int_{V_{\mathrm{II}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{II}})^{\mathrm{T}} \mathbf{D}_0^{\mathrm{II}} \boldsymbol{\varepsilon}_0^{\mathrm{II}} dV = \int_{V_{\mathrm{I}}} \tilde{\mathbf{u}}^{\mathrm{T}} \boldsymbol{\gamma}^{\mathrm{I}} dV + \int_{V_{\mathrm{II}}} \tilde{\mathbf{u}}^{\mathrm{T}} \boldsymbol{\gamma}^{\mathrm{II}} dV + \int_{S_2} \tilde{\mathbf{u}}^{\mathrm{T}} \mathbf{q} dS \quad \text{(S3)}$$

Thus Equation (S2) becomes

$$\int_{V_{\mathrm{I}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{I}})^{\mathrm{T}} \mathbf{D}_0^{\mathrm{I}} \Delta\boldsymbol{\varepsilon}^{\mathrm{I}} dV + \int_{V_{\mathrm{II}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{II}})^{\mathrm{T}} \mathbf{D}^{\mathrm{II}} \Delta\boldsymbol{\varepsilon}^{\mathrm{II}} dV = \int_{V_{\mathrm{II}}} (\tilde{\boldsymbol{\varepsilon}}^{\mathrm{II}})^{\mathrm{T}} \Delta\mathbf{D}^{\mathrm{II}} \boldsymbol{\varepsilon}_0^{\mathrm{II}} dV \quad \text{(S4)}$$

Equations (S3) and (S4) are equivalent to

$$\int_{V} \tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}} \mathbf{D}_0 \boldsymbol{\varepsilon}_0 dV = \int_{V} \tilde{\mathbf{u}}^{\mathrm{T}} \boldsymbol{\gamma} dV + \int_{S_2} \tilde{\mathbf{u}}^{\mathrm{T}} \mathbf{q} dS \quad \text{(S5)}$$

$$\int_{V} \tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}} \mathbf{D} \Delta\boldsymbol{\varepsilon} dV = \int_{V_{\mathrm{II}}} \tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}} \Delta\mathbf{D}^{\mathrm{II}} \boldsymbol{\varepsilon}_0^{\mathrm{II}} dV \quad \text{(S6)}$$

where $V = V_{\mathrm{I}} + V_{\mathrm{II}}$, $\mathbf{D}_0 = \mathbf{D}_0^{\mathrm{I}}$ and $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_0^{\mathrm{I}}$ and $\mathbf{D} = \mathbf{D}_0^{\mathrm{I}}$ and $\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\varepsilon}^{\mathrm{I}}$ outside the faults, $\mathbf{D}_0 = \mathbf{D}_0^{\mathrm{II}}$ and $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_0^{\mathrm{II}}$ and $\mathbf{D} = \mathbf{D}^{\mathrm{II}}$ and $\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\varepsilon}^{\mathrm{II}}$ inside the faults. Before the earthquake the initial fields (displacement, stress or strain) are solved from (S5). After an earthquake, the displacement change is solved from (S6) with the initial stress field from the strain field $\boldsymbol{\varepsilon}_0$. The total displacement and stress fields after the earthquake are obtained from the superposition of the initial fields and their changes.

Equations (S5) and (S6) are the theoretical framework of a new finite element model (NFEM) for studying earthquake triggering and stress transfer in a heterogeneous crust with multiple faults and initial stress field.

Suppose the region V is divided into m element outside the faults and n elements inside the faults. These elements construct a finite element system with p nodal points.

Formulae (S5) and (S6) can be expressed by

$$\sum_{e=1}^{m+n}\int_{V_e}(\tilde{\boldsymbol{\varepsilon}}_e)^{\mathrm{T}}\mathbf{D}_0^e\boldsymbol{\varepsilon}_0^e dV_e=\sum_{e=1}^{m+n}\Big(\int_{V_e}\tilde{\mathbf{u}}_e^T\boldsymbol{\gamma}_e dV_e+\int_{S_2^e}\tilde{\mathbf{u}}_e^T\mathbf{q}_e dS_e\Big) \tag{S7}$$

$$\sum_{e=1}^{m+n}\int_{V_e}(\tilde{\boldsymbol{\varepsilon}}_e)^T\mathbf{D}\Delta\boldsymbol{\varepsilon}^e dV_e=\sum_{e=1}^{n}\int_{V_e}(\tilde{\boldsymbol{\varepsilon}}_e)^T\Delta\mathbf{D}^{II}(\boldsymbol{\varepsilon}_0^e)^{II}dV_e \tag{S8}$$

The approximate displacement of $\mathbf{u}$ at an arbitrary point (ξ,η,ζ) inside an element e is interpolated from the approximate solutions $\mathbf{u}_e$ at the nodal points of the element; *i.e.*, $\mathbf{u}(\xi,\eta,\zeta)\approx\mathbf{H}(\xi,\eta,\zeta)\mathbf{u}_e$, here, $\mathbf{H}(\xi,\eta,\zeta)=[h_1\mathbf{I},\ h_2\mathbf{I},\ \ldots\ ,\ h_p\mathbf{I}]^{\mathrm{T}}$, and $\mathbf{I}$ is a 3×3 unit matrix, $h_i(\xi_j,\eta_j,\zeta_j)$ is the interpolation function or the shape function of the i th node of the element, which satisfies the conditions $\sum_{i=1}^{p}h_i=1$ where $h_i(\xi_j,\eta_j,\zeta_j)=1$ if $i=j$, otherwise $h_i(\xi_j,\eta_j,\zeta_j)=0$.

Introducing the differential operator $\mathbf{L}=[\mathbf{L_1}\ \ \mathbf{L_2}\ \ \mathbf{L_3}]$ and an element assembling matrix $\mathbf{S}_\mathrm{e}$ which connects the nodal displacement vector of the element and the global nodal displacement vector $\mathbf{U}$ of the finite element system [Cai, 1997], we have the nodal displacement vectors $\mathbf{u}_e=\mathbf{S}_e\mathbf{U}$. The strains in the element can be expressed in terms of the matrix $\mathbf{S}_e$ as $\boldsymbol{\varepsilon}(\xi,\eta,\eta)=\mathbf{Lu}\approx\mathbf{B}(\xi,\eta,\eta)\mathbf{u}_e=\mathbf{B}(\xi,\eta,\eta)\mathbf{S}_e\mathbf{U}$, here $\mathbf{B}=\mathbf{LH}$ and, $\mathbf{L}_1=[\partial/\partial x,\ 0,\ 0,\ 0,\ \partial/\partial z,\ \partial/\partial y]^{\mathrm{T}}$,

$\mathbf{L}_2=[0,\ \partial/\partial y,\ 0,\ \partial/\partial z,\ 0,\ \partial/\partial x]^{\mathrm{T}}$, $\mathbf{L}_3=[0,\ 0,\ \partial/\partial z,\ \partial/\partial y,\ \partial/\partial x,\ 0]^{\mathrm{T}}$. The virtual displacement $\tilde{\mathbf{u}}(\xi,\eta,\zeta)$ and strain $\tilde{\boldsymbol{\varepsilon}}(\xi,\eta,\zeta)$ in the element can also be approximated by its nodal virtual displacement as $\tilde{\mathbf{u}}\approx\mathbf{H}\tilde{\mathbf{u}}_e=\mathbf{HS}_e\tilde{\mathbf{U}}$ and

$\tilde{\boldsymbol{\varepsilon}} \approx \mathbf{B}\tilde{\mathbf{U}}_e = \mathbf{BHS}_e\tilde{\mathbf{U}}$. Inserting the approximate expressions of $\tilde{\mathbf{u}}, \tilde{\boldsymbol{\varepsilon}}$ and $\mathbf{u}, \boldsymbol{\varepsilon}$ into Formulae (S7) and (S8), we derive the finite element formulae to solve the global nodal displacement vector $\mathbf{U}_0$ before the earthquake and the global displacement change $\Delta\mathbf{U}$ induced by the earthquake as follows:

$$\mathbf{K}_0\mathbf{U}_0 = \mathbf{F}_0 \tag{S9}$$

$$\mathbf{K}\Delta\mathbf{U} = \Delta\mathbf{F} \tag{S10}$$

where

$$\mathbf{K}_0 = \sum_{e=1}^{m+n} \mathbf{S}_e^{\mathrm{T}} \left(\int_{V^e} \mathbf{B}^{\mathrm{T}}\mathbf{D}_0\mathbf{B}\, dV^e\right)\mathbf{S}_e, \tag{S11}$$

$$\mathbf{F}_0 = \sum_{e=1}^{m+n} \mathbf{S}_e^{\mathrm{T}} \left(\int_{V^e} \mathbf{H}^{\mathrm{T}}\boldsymbol{\gamma}\, dV^e + \int_{s_2^e} \mathbf{H}^{\mathrm{T}}\mathbf{q}\, dS_2^e\right), \tag{S12}$$

$$\mathbf{K} = \sum_{e=1}^{m+n} \mathbf{S}_e^{\mathrm{T}} \left(\int_{V^e} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, dV^e\right)\mathbf{S}_e, \tag{S13}$$

$$\Delta\mathbf{F} = \sum_{e=1}^{n} \mathbf{S}_e^{\mathrm{T}} \int_{V_{\mathrm{II}}^e} \mathbf{B}^{\mathrm{T}}\Delta\mathbf{D}\mathbf{B}\mathbf{U}_0^e\, dV_{\mathrm{II}}^e. \tag{S14}$$

Equations (S9) and (S10) build up a new FEM to model the evolution of displacement and stress for studying earthquake triggering in the earthquake sequence. From Formula (S14), it can be seen that $\Delta\mathbf{F}$ is dependent on both $\Delta\mathbf{D}$ and $\mathbf{U}_0^e$, so $\Delta\mathbf{U}$ in Formula (S10) is not only related to the changes of the material parameters $\Delta\mathbf{D}$ but also to the initial displacement $\mathbf{U}_0$ solved from Formula (S9). The total displacement field after the earthquake can be calculated by $\mathbf{U} = \mathbf{U}_0 + \Delta\mathbf{U}$, which will be taken as the initial displacement field for the next earthquake. The stress in an element can be calculated by using the nodal displacements of the element, *i.e.,* $\boldsymbol{\sigma}^e = \mathbf{D}\boldsymbol{\varepsilon}^e = \mathbf{DBu}^e$.

References

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